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ON UNIQUENESS OF POSITIVE RADIAL SOLUTIONS FOR POISSON EQUATION WITH NON-SMOOTH NONLINEARITIES

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Abstract

In this work, we derive a convolution-type Voltera integral equation for singular nonlinear equation and use the Schauder-Tychonoff fixed point theorem to proof existence and uniqueness of a fixed point of the resulting operator.

1. Introduction

This work is an improvement on existing *existence* and *uniqueness* results for positive radial solutions of the nonlinear Poisson equation

$$\Delta u + g(u) = 0,$$

having non-smooth nonlinearity g through equivalent ordinary differential equation

$$\frac{d^2u}{dr^2} + \frac{a}{r}\frac{du}{dr} = -g(u), \tag{1}$$

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$$r = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}, \tag{2}$$

$$a = n - 1, \ (n \geq 2). \tag{3}$$

Assuming that the nonlinearity $g : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following conditions:

(H1) $\xi g(\xi) > 0, \ \xi \neq 0,$

(H2) $g(0) = 0,$

(H3) $0 \leq \xi g(\xi) \leq k\xi^2, \ 0 < k < \infty,$

we will prove that there exists a unique positive decreasing solution to the Poisson equation. This constitutes an improvement to earlier works, where g is required to be differentiable. In [1], [6], and [13], continuity of g is used for only existence for boundary value problems; while in [12], $g \in C^1$ is assumed for the uniqueness of the following initial value problems:

$$\frac{d^2u}{dr^2} + \frac{a}{r}\frac{du}{dr} = -g(u),$$

$$g \in C([0, \infty)) \cap C^1(0, \infty), \ r \geq 0,$$

$$u(0) = \alpha, \tag{4}$$

$$u'(0) = 0. \tag{5}$$

In the current analysis, we relaxed the smoothness assumption on the nonlinearity g; as is observed in conditions (H1)-(H3). Our uniqueness result is significant in the sense that, we do not impose any smoothness assumption on the nonlinearity g. We gave illustrative example to which our result applies, in the last section of this paper.

Our existence result is based on an application of the Schauder-Tychonoff theorem stated below (see, for example, Corduneanu [4], O'regan [11]).

Theorem 1.1 (Schauder-Tychonoff). *Let E be a locally convex Hausdorff space and let T be a continuous mapping from a convex subset $K \subset E$ into E such that $T(K) \subset A \subset K$, with A compact. Then there exists a fixed point for T.*

In the sequel, we shall make use of the following spaces defined below:

Definition 1.1 (Function spaces). The space of all continuous real-valued functions from $\mathbb{R}$ into $\mathbb{R}$ is denoted by $C(\mathbb{R}, \mathbb{R})$, and the restriction of elements of $C(\mathbb{R}, \mathbb{R})$ to the interval $[0, \tau]$ is denoted by $C([0, \tau], \mathbb{R})$, while C_0 denotes the space of functions $u(t)$ in $C(\mathbb{R}, \mathbb{R})$ satisfying $\lim_{t\to\infty} u(t) = 0$.

2. Preliminaries

Proposition 2.1. *Any positive solution to the initial value problems* (1), (4), *and* (5) *has its only extremum at the origin for all* $r \in [0, \infty)$.

Proof. Multiply (1) by $r^a (a \in \mathbb{N},\ a \geq 1)$ to obtain $(r^a u')' = -r^a g(u)$ which, on integration, yields

$$r^a u' = -\int_0^r \sigma^a g(u(\sigma)) d\sigma$$

$$\Rightarrow u'(r) = -\frac{\int_0^r \sigma^a g(u(\sigma)) d\sigma}{r^a}. \tag{6}$$

Setting $g(u(r)) = \rho(r)$ for some function ρ, (6) becomes $u'(r) = -\frac{\int_0^r \sigma^a \rho(\sigma) d\sigma}{r^a}$. Since $r^a \geq 0$, by the mean value theorem, we have for some $\theta\,(0 < \theta < r)$

$$u' = -\frac{\rho(\theta) r}{a+1}, \tag{7}$$

so that $u'(0) = 0$ and since $r \geq 0$, we have that the sign of $u'(r)$ does not change in $[0, \infty)$.

Proposition 2.2. *Let* (H1) *holds. Then all solutions to the initial value problems* (1), (4), *and* (5) *are positive and decreasing, if and only if* $g(\alpha) > 0$.

Proof. Let $u = u(r)$ be a positive and decreasing solution of the initial value problems (1), (4), and (5), then obviously by Proposition 2.1, the only extremum $u(0) = \alpha$ must be positive and by condition H1, $g(\alpha) > 0$.

On the other hand, we assume that $g(u(0)) = g(\alpha) > 0$ and prove that $u(r)$ is positive and decreasing. Set $g(u(r)) = \rho(r)$ for some continuous function $\rho : \mathbb{R}_+ \to [0, \sup_{\xi \in \mathbb{R}} g(\xi)]$. If $u(0) > 0$, then by the condition H1, $ug(u) > 0$. Which implies $u(0)g(u(0)) = \alpha g(\alpha) > 0$ yields $\rho(0) > 0$. It follows from the continuity of ρ that $\rho(\theta_0) > 0$ for some $\theta_0 > 0$ in some neighborhood of 0. Applying (7), we observe that $u'(r)$ is negative in such a neighborhood, which implies that $u(r)$ is strictly decreasing from $u(\alpha)$ in the immediate neighborhood of 0. We shall use further application of (7) to show that $u'(r) < 0$ for all r.

If there are two points θ_1 and θ_2 in (θ_0, ∞) such that $\rho(\theta_1) = \rho(\theta_2)$, $\theta_1 \neq \theta_2$, then (7) gives $u'(r_1) = -\dfrac{\rho(\theta_1) r_1}{a+1}$, $\theta_0 < \theta_1 < r_1$ and $u'(r_2) = -\dfrac{\rho(\theta_2) r_2}{a+1}$, $\theta_0 < \theta_2 < r_2$, where $r_1 < r_2$ and we have $u'(r_1) > u'(r_2)$, which shows $u(r)$ is strictly decreasing for all $r \in (\theta_0, \infty)$. And finally, if $\rho(\theta_1) \neq \rho(\theta_2)$ for $\theta_1 \neq \theta_2$. Then $\phi(\theta) > 0$ for all $\theta \in (\theta_0, \infty)$, which implies $u'(r)$ does not change sign. This implies that $u'(r) < 0$ for all $r \in [\theta_0, \infty)$, therefore $u(r)$ is a positive and decreasing function.

Theorem 2.1. *Let u satisfy the initial value problems* (1), (4), *and* (5) *and the function* $v(r)$ *be given by the integral identity*

$$v(r) = \frac{a\int_0^r \sigma^{a-1}u(\sigma)d\sigma - \int_0^r (r-\sigma)\sigma^a g(u(\sigma))d\sigma}{r^a}. \tag{8}$$

Then $v(r)$ satisfies the initial value problem, if and only if

$$\lim_{r\downarrow 0} v(r) = \alpha,$$

$$\lim_{r\downarrow 0} v'(r) = 0.$$

Proof. Suppose (8) holds. We shall show that $\lim_{r\downarrow 0} v(r) = \alpha$ and $\lim_{r\downarrow 0} v'(r) = 0$. We observe that by the initial condition (5) $u(r)$ has one-sided derivative at the origin, hence it is continuous at the origin; so that $\lim_{r\downarrow 0} u(r) = u(0) = \alpha$. Further, we may take the limit of (1) to deduce $\lim_{r\downarrow 0} u''(r) = -\frac{g(\alpha)}{a+1}$; so that $\lim_{r\downarrow 0} u'(r) = -\lim_{r\downarrow 0} \frac{r}{a}(u''(r) + g(u)) = 0$. Hence, using (8),

$$\lim_{r\downarrow 0} v(r) = \lim_{r\downarrow 0}\left[\frac{a\int_0^r \sigma^{a-1}u(\sigma)d\sigma - \int_0^r (r-\sigma)\sigma^a g(u(\sigma))d\sigma}{r^a}\right]$$

$$= \lim_{r\downarrow 0}\left[\frac{\frac{d}{dr}\{a\int_0^r \sigma^{a-1}u(\sigma)d\sigma - \int_0^r (r-\sigma)\sigma^a g(u(\sigma))d\sigma\}}{ar^{a-1}}\right]$$

(by L'Hopital's rule)

$$= \lim_{r\downarrow 0}\left[\frac{ar^{a-1}u(r) - \int_0^r \sigma^a g(u(\sigma))d\sigma}{ar^{a-1}}\right] \text{ (by L'Hopital's rule)}$$

$$= \lim_{r\downarrow 0}\left[\frac{ar^{a-1}u(r)}{ar^{a-1}}\right] - \lim_{r\downarrow 0}\left[\frac{r^a g(u(r))}{a^2 r^{a-2}}\right]$$

$$= \lim_{r\downarrow 0} u(r) - \lim_{r\downarrow 0}\left[\frac{r^2 g(u(r))}{a^2}\right]$$

$$= \lim_{r\downarrow 0} u(r)$$

$$\Rightarrow \lim_{r\downarrow 0} v(r) = u(0) = \alpha.$$

By differentiating (8) and taking limits, we similarly deduce

$$\lim_{r\downarrow 0} v'(r) = \left(1 - \frac{a^2}{a+1}\right) u'(0) = 0.$$

On the other hand, we assume that $\lim_{r\downarrow 0} v(r) = \alpha$ and $\lim_{r\downarrow 0} v(r) = 0$; and prove that (8) gives a solution to the initial value problems (1), (4), and (5). It suffices to show that $v(r)$ satisfies the differential equation (1). We claim that v is identically equal to u. To proof this claim, we proceed as follows:

$$r^a v(r) - a\int_0^r \sigma^{a-1} u(\sigma) d\sigma = -\int_0^r (r-\sigma)\sigma^a g(u(\sigma)) d\sigma$$

$$\Rightarrow (r^a v(r) - r^a u(r)) + \int_0^r \sigma^a u'(\sigma) d\sigma = -\int_0^r \int_0^\mu \sigma^a g(u(\sigma)) d\sigma d\mu$$

$$\Rightarrow \frac{d}{dr}(r^a v(r) - r^a u(r)) + r^a u'(r) = -\int_0^r \sigma^a g(u(\sigma)) d\sigma$$

$$\Rightarrow \frac{d^2}{dr^2}(r^a v(r) - r^a u(r)) + \frac{d}{dr}[r^a u'(r)] = -r^a g(u(r))$$

$$\Rightarrow \frac{d^2}{dr^2}(r^a v(r) - r^a u(r)) = 0, \text{ (since } u \text{ solves (1))}. \tag{9}$$

Integrating (9), we get

$$(r^a(v-u))' = r^a(v'-u') + ar^{a-1}(v-u) = b = \text{constant}; \tag{10}$$

from whence taking limits as $r \downarrow 0$, we deduce that $b = 0$. Integrating (10), we get

$$r^a(v-u)=c, \tag{11}$$

from whence we also deduce that $c = 0$, after taken limits as $r \downarrow 0$. Thus, v is identically equal to u.

3. Main Results

The results obtained in Section 2 assumed existence of solution to the initial value problem; i.e., their validity is consequent upon existence of solutions. The next step is to establish existence of solution; and it involves showing that (8) induces a compact operator T from the space $C(\mathbb{R}_+, \mathbb{R})$ into itself, the fixed point of which is $u(r)$. We shall show that the image of T on any restriction of elements of $C(\mathbb{R}, \mathbb{R})$ to $C([0, \tau], \mathbb{R})$ is relatively compact in $C(\mathbb{R}_+, \mathbb{R})$. For relative compactness, we require that sequences in $T[C([0, \tau], \mathbb{R})]$ be equibounded and equicontinuous (Yosida [14]).

Theorem 3.1. *Let g satisfy the conditions* (H1)-(H3), *then the initial value problem arising from Equations* (1), (4), *and* (5) *have a solution.*

Proof. A little modification on

$$r^a u(r) = a\int_0^r \sigma^{a-1}u(\sigma)d\sigma - \int_0^r (r-\sigma)\sigma^a g(u(\sigma))d\sigma,$$

yields the convolution operator;

$$T(w) = a\int_0^r (r+1-\sigma)\frac{w}{\sigma}\,d\sigma - \int_0^r (r-\sigma)\left[\sigma^a g\left(\frac{w}{\sigma^a}\right) + a\,\frac{w}{\sigma}\right]d\sigma,$$

that is, $Tw = L_1 g_1(r, w) - L_2 g_2(r, w)$ by setting $g_1(r, w) = \frac{aw}{r}$ and $g_2(r, w) = r^a g\left(\frac{w}{r^a}\right) + ag_1(r, w)$, where $w(r) = r^a u(r)$.

We desire to apply Schauder-Tychonoff theorem Theorem 1.1 to show that w is a fixed point of T. Any bounded sequence of functions $\{\sigma_n\} \subseteq C_0$ is equibounded and equicontinuous on $[0, \alpha)$ and by Arzela-Ascoli theorem such sequences are relatively compact. Now, since T is a continuous operator, it maps relatively compact sets into relatively compact sets, therefore we infer that T is a compact operator so T maps every closed convex set K into itself. Further, (see, for example, Corduneanu [4] and O'regan [11]) arbitrary restrictions above the image of T on $C(\mathbb{R}_+, \mathbb{R})$ is relatively compact. Therefore, T induces a compact operator T from the space $C(\mathbb{R}_+, \mathbb{R})$ of continuous functions into itself the fixed point of which is $r^a u(r)$, since the image of T on any restriction of elements of $C(\mathbb{R}, \mathbb{R})$ to $C([0, \tau], \mathbb{R})$ is relatively compact in $C(\mathbb{R}_+, \mathbb{R})$. Therefore by Schauder-Tychonoff theorem, $r^a u(r)$ is a fixed point of T given by:

$$(Tr^a u)(r) = a\int_0^r (r + 1 - \sigma)\sigma^{a-1}u(\sigma)d\sigma - \int_0^r (r - \sigma)[\sigma^a g(u) + a\sigma^{a-1}u]d\sigma,$$

and the solution $u(r)$ of the initial value problem exists and is well defined by $u(r) = \dfrac{(Tr^a u(r))(r)}{r^a}$.

Proposition 3.1. *Let g be a continuous function satisfying $\xi g(\xi) > 0$, for $\xi \neq 0$ and $g(0) = 0$; $0 \leq \xi \leq \alpha$. Then if $u(r)$ is a strictly decreasing positive function with values between zero and α satisfying the initial value problem*:

$$\frac{d^2u}{dr^2}(r) + \frac{a}{r}\frac{du}{dr}(r) = -g(u),$$

$$u(0) = \alpha,$$

$$\frac{du}{dr}(0) = 0.$$

Then $g(u(r))$ is positive and strictly decreasing with increasing r.

Proof. That g is positive follows directly from the condition (H1); $u(r)g(u(r)) > 0$ and Proposition 2.2. To prove that $g(u(r))$ is strictly decreasing with r, we divide the condition (H3) by the positive function $u(t)$ to obtain $0 \leq g(u(r)) \leq ku(r)$ and observe that $g(u(r))$ is strictly decreasing with increasing r.

Theorem 3.2. *The solution to the problems* (1), (4), *and* (5) *are unique for all continuous nonlinearities* $g(u(t))$ *satisfying* (H1)-(H3).

Proof. Let $u(r)$ and $v(r)$ be two distinct solutions of the problem. Since both solutions are, by Proposition 3.1, positive and strictly decreasing, there exists intervals in $[0, \infty)$ such that either $u(r) < v(r)$ or $u(r) > v(r)$. We assume, without loss of generality, that $u(r) < v(r)$ in some interval $[0, r_0] \subset [0, \infty)$. If $u(r)$ and $v(r)$ are such two distinct solutions of the initial value problem, then for all $r \in [0, r_0]$, we have that;

$$r^a(u(r) - v(r)) = a\int_0^r (r + 1 - \sigma)\sigma^{a-1}[u(\sigma) - v(\sigma)]d\sigma$$

$$- \int_0^r (r - \sigma)[\sigma^a[g(u(\sigma)) - g(v(\sigma))] + \sigma^{a-1}[u(\sigma) - v(\sigma)]]d\sigma.$$

Setting $u(r) - v(r) = z(r)$ and $g(u(r)) - g(v(r)) = b(r)$, we consider the initial value problem

$$\frac{d^2 z}{dr^2} = \frac{-a}{r}\frac{dz}{dr} - b(r),$$

$$z(0) = 0,$$

$$z'(0) = 0,$$

which must have in the interval $[0, r_0]$, the solution $r^a z(r) = a\int_0^r (r + 1 - \sigma)\sigma^{a-1}z(\sigma)d\sigma - \int_0^r (r - \sigma)[\sigma^a b(\sigma) + \sigma^{a-1}z(\sigma)]d\sigma.$ Now, in Proposition 2.2, it is shown that $u(r)$ and $v(r)$ are positive and strictly decreasing solutions. This means that $z(r)$ must be bounded since $\lim_{r\to r_0} z(r) = 0$ and $z(0) = 0$. This implies that there exists a point r_1 in $(0, r_0)$ with $u'(r_1) = v'(r_1)$. We shall end the proof by showing that this contradicts the assumption that; $u(r) < v(r)$ in $[0, r_0]$.

If $z'(r_1) = 0,$ we must have $u'(r_1) - v'(r_1) = 0,$ which yields

$$\Rightarrow r_1^a (u - v)'(r_1) = -a\int_0^{r_1} (r - \sigma)\sigma^a [g(u(\sigma)) - g(v(\sigma))]d\sigma$$

$$\Rightarrow a\int_0^{r_1} (r - \sigma)\sigma^a g(u(\sigma))d\sigma = a\int_0^{r_1} (r - \sigma)\sigma^a g(v(\sigma))d\sigma$$

$$\Rightarrow g(u(\vartheta_u))\int_0^{r_1} (r - \sigma)\sigma^a d\sigma = g(v(\vartheta_v))\int_0^{r_1} (r - \sigma)\sigma^a d\sigma,$$

(by mean value theorem)

where ϑ_u and ϑ_v are distinct points in the interval $(0, r_0)$. By Proposition 3.1, g is strictly decreasing and so $g(u(r)) < g(v(r)),$ whenever $u(r) < v(r).$ It follows from above that, the equation $g(u(\vartheta_u)) = g(v(\vartheta_v))$ gives $u(\vartheta_u) = v(\vartheta_v)$ (where $\vartheta_u, \vartheta_v \in (0, r_0) \subset [0, r_0]$) yields $u(\vartheta_u) = v(\vartheta_v).$ This contradicts the fact that $u(r) < v(r)$ for all $r \in [0, r_1) \subset [0, r_0]$ for some $r_1 \in (0, r_0).$

Further, suppose $u(r) < v(r) \forall r,$ we observe that in the limit as $r_0 \to \infty$ $u(r_0) \to 0$ and so we must have $\vartheta_u, \vartheta_v \in (0, \infty)$ and by the argument above, we obtain $u(\vartheta_u) = v(\vartheta_v),$ which contradicts the assumption that $u(r) < v(r) \forall r \in (0, \infty).$ Therefore, the solution is unique.

4. Illustrative Example

For any integer $m \geq 1,$ consider the initial value problem

$$\frac{d^2u}{dr^2} + \frac{a}{r}\frac{du}{dr} = -g(u) := -\begin{cases} u^{\frac{1}{2m+1}}, & \text{if } -\infty \leq u \leq -1 \\ u^{2m+1}, & \text{if } -1 \leq u \leq 1 \\ u^{\frac{1}{2m+1}}, & \text{if } 1 \leq u \leq \infty. \end{cases} \tag{12}$$

$$u(0) = \alpha, \tag{13}$$

$$u'(0) = 0. \tag{14}$$

The given non-smooth nonlinearity $g(u)$ satisfies the conditions H1-H3, so that Theorems 3.1 and 3.2 guarantee the existence of a unique solution to the initial value problems (12)-(14).

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